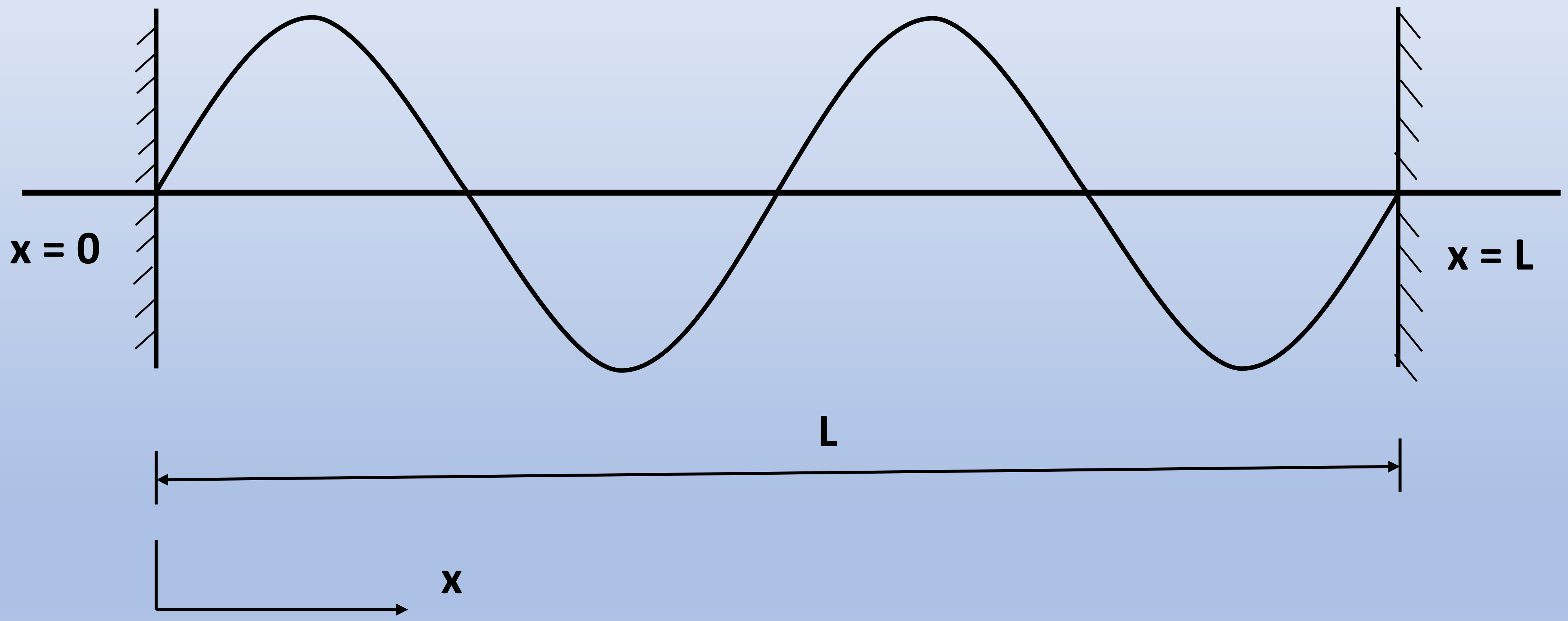
Solve 1D Wave Equation Using Analytical Methods

Objectives

- Present a simple 1D Vibration problem, which is vibrations of an elastic string
- Solve the problem, which is 1D wave equation, using Analytical methods.
- The analytical methods we present are based on separation of variables method and D'Alembert's method.

1D Wave Equation Problem – Vibrations of an elastic string



1D Wave Equation

Assumptions and conditions :

- Thin flexible string with negligible weight
- The two ends of the string are clamped, so the displacements are zero at the ends
- There is no damping involved
- Initial displacement profile is given
- Initial velocity is considered zero

1D Wave Equation

- $u_{tt} = c^2 * u_{xx}$ ....(1);
- u(x, t); u – vertical displacement, c – wave propagation speed
- x – spatial coordinate, t – time; 0 <= x <= L; t > 0;
- $c^2 = \left(\frac{T}{\rho}\right)$;
- T – force of tension exerted on the string ; ρ - mass density (mass per unit length of the string)
- Boundary Conditions:
- u(0,t) = 0; u(L,t) = 0;
- Initial Conditions:
- u(x,0) = f(x) ; u_t (x,0) = g(x)

1D Wave Equation – Solution By Separation of Variables Method

- Simplified Case : f(x) ≠ 0; g(x) = 0;
- i.e., Initial Conditions: u(x,0) = f(x); u_t (x,0) = 0;
- $u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{cn\pi t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$
- $A_n = \left(\frac{2}{L}\right) \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$ n = 1,2,3….
- e.g., f(x) = $c1*\sin(c2 * \pi x)$

1D Wave Equation – Solution By Separation of Variables Method

Particular Solution:

- $u(x, 0) = f(x) = c1 * \sin(c2 * \pi x)$ is in the form of Fourier sine series
- $A_n = \left(\frac{2}{L}\right) \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$ n = 1,2,3….
- $A_n = \left(\frac{2}{L}\right) \int_0^L c1 * \sin(c2 * \pi x) * \sin\left(\frac{n\pi x}{L}\right) dx$ n = 1,2,3….
- $A_n = c1$ for, $n = c2 * L$; (Here the arguments of both sin functions are the same)
- $A_n = 0$ for all other $n, n \neq c2 * L$
- $u(x, t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{cn\pi t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$
- $u(x, t) = c1 * \cos(c * c2 * \pi * t) \sin(c2 * \pi * x)$………………………………………..(2)

1D Wave Equation – Solution By Separation of Variables Method

Particular Solution:

- Example:
- $u(x, t) = c1 * \cos(c * c2 * \pi * t) \sin(c2 * \pi * x)$…………………………………….(2)
- If c1 = 1, c2 = 2, c = 0.5, f(x) = $\sin(2 * \pi * x)$
- $u(x, t) = \cos(1 * \pi * t) \sin(2 * \pi * x)$

1D Wave Equation – Solution By D'Alembert's Method

- $u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{cn\pi t}{L}\right)$;

[Analytical solution obtained using Sep of Variables Method]

- Using the product rule, $\sin(A)\cos(B) = \left(\frac{1}{2}\right)(\sin(A - B) + \sin(A + B))$

- $u(x,t) = \left(\frac{1}{2}\right)\sum_{n=1}^{\infty} A_n \left(\sin\left(\frac{n\pi(x - ct)}{L}\right) + \sin\left(\frac{n\pi(x+ct)}{L}\right)\right)$

- $A_n = \left(\frac{2}{L}\right)\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ n = 1,2,3....

- $A_n = c1$ for, $n = c2 * L$; (Here the arguments of both sin functions are the same)

- $A_n = 0$ for all other $n, n \neq c2 * L$

1D Wave Equation – Solution By D'Alembert's Method

- $\mathrm{u(x,t)} = \left(\frac{1}{2}\right)\left(\mathrm{c1}*\sin\left(\mathrm{c2}*\pi \; *(x-ct)\right)+\mathrm{c1}*\sin\left(\mathrm{c2}*\pi \; *(x+ct)\right)\right)$
- $\mathrm{u(x,t)} = \left(\frac{1}{2}\right)(\mathrm{F}(x \; - ct)+\mathrm{F}(x+ct))$ ………………………………………………(3)
- F(x) is the odd periodic extension (period 2L) of the initial displacement f (x).
- Example:
- If c1 = 1, c2 = 2, c = 0.5, f(x) = $\sin(2 \; * \; \pi * \mathrm{x})$
- $\mathrm{u(x,t)} = \left(\frac{1}{2}\right)\left(\sin\left(2*\pi \; *(x-0.5t)\right)+\sin\left(2*\pi \; *(x+0.5t)\right)\right)$

Summary

In this video,

- We presented a 1D wave equation that describes the vibration of an elastic string
- We solved the problem using 2 different analytical methods.
- The analytical solutions we used are obtained using

  a) Separation of variables method

  b) D’Alembert’s method

- In future videos, we can explore more challenging problems.