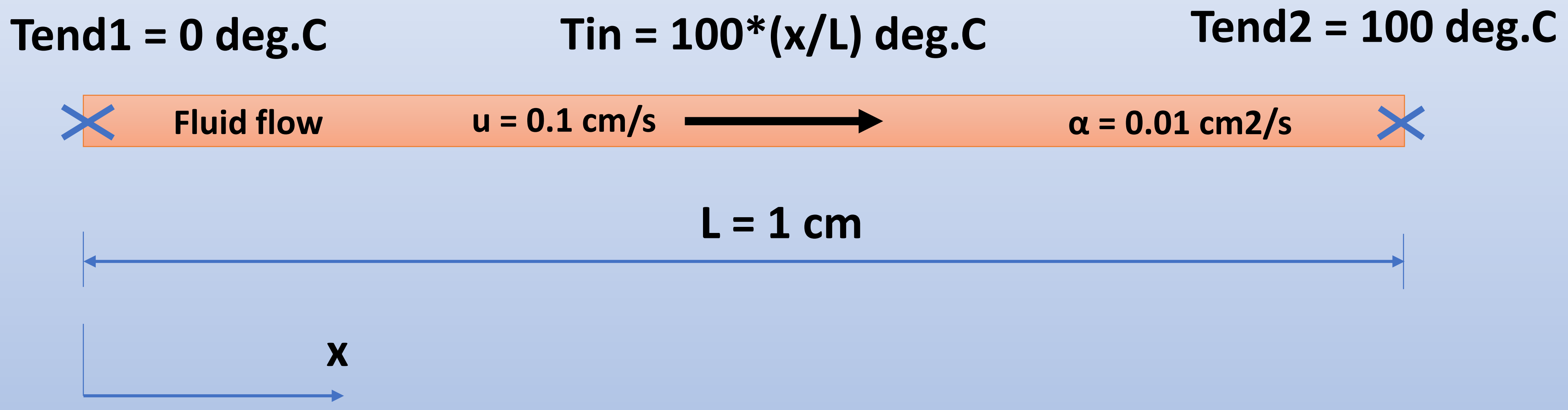
Solve 1D Transient Advection-Diffusion Problem Using FTCS Finite Difference Method

Objectives

- Present a simple 1D Transient Advection-Diffusion problem
- Similar to the diffusion problem we have seen earlier
- We discretize the domain into 5 grid spacings
- We consider a single time step and solve the problem using FTCS FDM
- Vary grid spacings and time steps and obtain solutions

1D Advection-Diffusion Problem



Porous Plate

1D Advection-Diffusion Equation

- $f_t + u * f_x = \alpha * f_{xx}$ ....(1); f(x, t);   u – convection velocity, α – diffusion coefficient
- In terms of temperature, $f(x, t) = T(x, t)$
- $T_t + u * T_x = \alpha * T_{xx}$
- $\frac{\partial T}{\partial t} + u * \frac{\partial T}{\partial x} = \alpha * \frac{\partial^2 T}{\partial x^2}$ ....(2); T(Temp) = T(x, t); x – spatial coordinate, t – time
- α $- Thermal\ diffusivity, \frac{cm2}{s}$; $\alpha = k/(\rho * c)$
- k – thermal conductivity of the fluid , W/(cm K) (assumed constant)
- ρ – density of the fluid, kg/cm3
- c – specific heat capacity of the fluid, J/(kg K)
- km – thermal conductivity of the material (neglected compared with that of the fluid), W/(cm K)

1D Advection-Diffusion Equation – Exact Solution – At Steady State

- $T(x,\infty) = 100 * \frac{\exp^{\left(\frac{(P*x)}{L}\right)} - 1}{\exp^{P} - 1}$; ……………………………………….(3)
- Peclet No. $P = \left(\frac{u * L}{\propto}\right)$

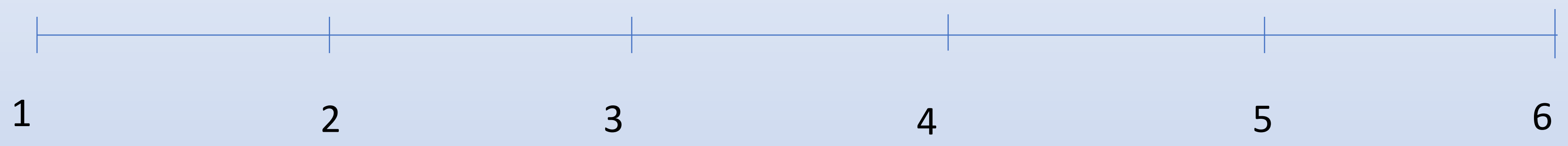
FTCS - Forward-Time Centered-Space Method

- $T_t + u * T_x = \alpha * T_{xx}$
- Let us discretize the domain. Here, i represents the node location and n represents the time step.
- $\left(\frac{T_i^{n+1} - T_i^n}{\Delta t}\right) + u * \left(\frac{T_{i+1}^n - T_{i-1}^n}{2 * \Delta x}\right) = \alpha * \left(\frac{T_{i-1}^n - 2 * T_i^n + T_{i+1}^n}{\Delta x^2}\right)$; Rearranging, we get
- $T_i^{n+1} = T_i^n - \left(\frac{u * \Delta t}{2 * \Delta x}\right) * (T_{i+1}^n - T_{i-1}^n) + (\frac{\alpha * \Delta t}{\Delta x^2}) * (T_{i-1}^n - 2 * T_i^n + T_{i+1}^n)$;.........(4)
- Let c = $(\frac{u * \Delta t}{\Delta x})$; where c = courant/convection number;
- Let d = $(\frac{\alpha * \Delta t}{\Delta x^2})$; where d – diffusion number, we then get
- $T_i^{n+1} = T_i^n - \left(\frac{c}{2}\right) * (T_{i+1}^n - T_{i-1}^n) + d * (T_{i-1}^n - 2 * T_i^n + T_{i+1}^n)$;.........(5)
- Equation (5) is the FTCS finite difference approximation of the original equation

FTCS - Forward-Time Centered-Space Method

- FTCS method is an explicit method and is conditionally stable.
- Stability criteria : $c^2 <= 2*d <= 1$
- The error is $O(\Delta t) + O(\Delta x^2)$

- Now let us discretize the 1D domain into, m (say 5) segments / grid spacings (equally spaced) as shown below.



- Note, temperatures at Node 1) and Node 6) are known (BCs).
- To apply equation (5), we need to consider the interior nodes 2 to 5.

- $T_i^{n+1} = T_i^n - \left(\frac{c}{2}\right) * (T_{i+1}^n - T_{i-1}^n) + d * (T_{i-1}^n - 2 * T_i^n + T_{i+1}^n)$;.........(5)
- Let i = 2, 3, 4 & 5 & n = 0, Eq 5) becomes
- $T_2^1 = T_2^0 - \left(\frac{c}{2}\right) * (T_3^0 - T_1^0) + d * (T_1^0 - 2 * T_2^0 + T_3^0)$;.........(6)
- $T_3^1 = T_3^0 - \left(\frac{c}{2}\right) * (T_4^0 - T_2^0) + d * (T_2^0 - 2 * T_3^0 + T_4^0)$;.........(7)
- $T_4^1 = T_4^0 - \left(\frac{c}{2}\right) * (T_5^0 - T_3^0) + d * (T_3^0 - 2 * T_4^0 + T_5^0)$;.........(8)
- $T_5^1 = T_5^0 - \left(\frac{c}{2}\right) * (T_6^0 - T_4^0) + d * (T_4^0 - 2 * T_5^0 + T_6^0)$;.........(9)
- Here T_1^n, T_6^n are the BC's for all times and T_i^0 is the initial condition (IC) for all nodes.

- Let $\Delta t = 0.5$ s; $\Delta x = L/m = 1/5 = 0.2$ cm
- Stability criteria : $c^2 <= 2*d <= 1$
- Stability Criteria I
- $d = \left(\frac{\alpha * \Delta t}{\Delta x^2}\right) = \left(\frac{0.01 \text{ x } 0.5}{0.2^2}\right) = 0.125;$
- $2*d = 0.25 <= 1$ [Stability Criteria I met]
- Stability Criteria II
- $c = \left(\frac{u * \Delta t}{\Delta x}\right) = \left(\frac{0.1 \text{ x } 0.5}{0.2}\right) = 0.25$
- $c^2 = 0.25^2 = 0.0625 <= 2*d <= 0.25$ [Stability Criteria II met]

- Initial Conditions:
- $T(x,0) = 100 * \left(\frac{x}{L}\right)$; x = 0.0, 0.2, 0.4, 0.6, 0.8, 1; (for six nodes)
- T(0,0) = 0 °C; T(0.2,0) = 20 °C; T(0.4,0) = 40 °C;
- T(0.6,0) = 60 °C; T(0.8,0) = 80 °C; T(1,0) = 100 °C;

- Boundary Conditions:
- T(0,t) = 0 °C;
- T(L,t) = 100 °C;

- We need to substitute the values of c, d and the ICs, BCs into equations (6) to (9)

- Substituting the initial and boundary conditions into equations (6) to (9)
- T_2^1 = 20 - $\left(\frac{0.25}{2}\right) * (40 \ - 0) +$ 0.125 * $(0 - 2 * 20 + 40)$;.........(6)
- T_3^1 = 40 - $\left(\frac{0.25}{2}\right) * (60 \ - 20)$ + 0.125 * $(20 - 2 * 40 + 60)$;.........(7)
- T_4^1 = 60 - $\left(\frac{0.25}{2}\right) * (80 \ - 40)$ + 0.125 * $(40 - 2 * 60 + 80)$;.........(8)
- T_5^1 = 80 - $\left(\frac{0.25}{2}\right) * (100 \ - 60)$ + 0.125 * $(60 - 2 * 80 + 100)$;.........(9)
- $T_2^1 = 15$ °C
- $T_3^1 = 35$ °C
- $T_4^1 = 55$ °C
- $T_5^1 = 75$ °C

- Likewise, we can find the temperatures at these interior nodes at the next time step by choosing $n = 2$ and so on

- Graphical results are presented using MATLAB for this case.

- Using MATLAB or other software, we can develop codes for a general case where the number of grid spacings and time steps can be varied as desired.

- And the solutions obtained accordingly.

Summary

In this video,

- We presented a 1D Advection-Diffusion Problem
- We discretized our domain and solved the problem using FTCS Finite difference method
- We varied the grid spacings and time steps and presented the results using MATLAB
- In future videos, we can explore more challenging problems.